

THE HODGE NUMBER $h^{1,1}$ OF IRREGULAR ALGEBRAIC SURFACES

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ABSTRACT. We prove a new inequality for the Hodge number $h^{1,1}$ of irregular complex smooth projective surfaces of general type without irregular pencils of genus ≥ 2 . More specifically we show that if the irregularity q satisfies $q = 2^k + 1$ then $h^{1,1} \geq 4q - 3$. This generalizes results previously known for $q = 3$ and $q = 5$.

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1. INTRODUCTION

Let S be a complex smooth projective surface of general type. We let $h^{p,s} = \dim H^{p,s}(S)$ be its (p, s) Hodge number. One has that $h^{1,0}$ is the irregularity q of S whilst $h^{2,0}$ is its geometric genus p_g . We say that S has an irregular pencil of genus b if there is a morphism with connected fibres $\pi : S \rightarrow B$ over a curve B of genus $b > 0$.

In this paper we prove the following theorem:

Theorem 1.1. *If S is a complex smooth projective surface of general type without irregular pencils of genus ≥ 2 and $q = 2^k + 1$ then $h^{1,1} \geq 4q - 3$.*

We note that $h^{1,1} \geq 3q - 2$ for complex smooth projective surfaces of general type. In fact from the Bogomolov-Myiaoka-Yau inequality $c_2 \geq 3\chi$, and the fact that $c_2 = 2 - 4q + 2p_g + h^{1,1}$ one obtains

$$(1) \quad h^{1,1} \geq p_g + q + 1.$$

Then, by [3], $p_g \geq 2q - 4$ and equality holds if and only if S is birational to a product of two curves one of which has genus 2. In this last case, one has $c_2 = 4\chi$, and $p_g = 2(q - 2)$ hence $h^{1,1} = 4q - 6$. Since in this case also $q \geq 4$, we conclude that always

$$(2) \quad h^{1,1} \geq 3q - 2.$$

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For surfaces S without irregular pencils of genus ≥ 2 , Lazarsfeld and Popa in [11] reproved inequality (2) and also that, if q is odd,

$$(3) \quad h^{1,1} \geq 3q - 1.$$

If there is an irregular pencil with connected fibres $\pi : S \rightarrow B$ over a curve B of genus b , the following inequality has been recently proven in [12]:

$$(4) \quad h^{1,1} \geq 2b(q - b) + 2 + \sum (l(F) - 1)$$

where $l(F)$ is the number of the irreducible components of a fiber F .

In some instances, for surfaces S without irregular pencils of genus ≥ 2 , inequalities (2) and (3) can be improved:

- in the case $q = 3$, $h^{1,1} \geq 9$ by [5]. No examples of surfaces with $h^{1,1} = 9$ are known; the symmetric product of a curve of genus 3, however, has $h^{1,1} = 10$;
- when $q = 4$, in [5] it is proven that $h^{1,1} \geq 11$. The Schoen surface (see [16] and also [7]) realizes the smallest known value of $h^{1,1} = 12$; the next known example, $h^{1,1} = 17$, is given by the symmetric product of a curve of genus 4;
- for $q = 5$ we have $h^{1,1} \geq 17$ again by [5]. No examples are known for $h^{1,1} < 25$; the Fano surface of the lines of a smooth cubic 3-fold has $h^{1,1} = 25$; the symmetric product of a curve of genus 5 gives $h^{1,1} = 26$.

Theorem 1.1 extends the above cases $q = 3$ and $q = 5$. To prove Theorem 1.1, we use the the following linear algebra result to estimate the dimension of the $(1,1)$ part of image of the cup product map $\bigwedge^2 H^1(S, \mathbb{C}) \rightarrow H^2(S, \mathbb{C})$:

Proposition 1.2. *Let \mathcal{H} be the real vector space of $q \times q$ complex hermitian matrices.*

If $q = 2^k + 1, k \geq 2$ and $L \subset \mathcal{H}$ is a (real) vector subspace such that every $X \in L \setminus \{0\}$ has at least 2 positive and 2 negative eigenvalues then $\dim L \leq q^2 - (4q - 3)$.

Theorem 1.1 does not hold for the Schoen surface which has $q \neq 2^k + 1$. We notice the analogy with the role played by 2-adic valuations for the estimates of dimensions of spaces of constant rank matrices in [1] and [6].

The paper is organized as follows. In Section 2 we explain how hermitian matrices relate to lower bounds for $h^{1,1}$ and in Section 3 we prove Proposition 1.2 and Theorem 1.1. Finally in Section 4 we give some corollaries of Theorem 1.1.

2. ALBANESE VARIETY AND HERMITIAN MATRICES

Given a complex variety V with a real structure, we denote by $V(\mathbb{R})$ (or by $V_{\mathbb{R}}$, should the notation become too cumbersome) the locus of its real points.

Given a complex smooth projective irregular surface of general type S , let A be the Albanese variety of S and $a : S \rightarrow A$ the Albanese mapping. Defining the pull-back map

$$a^* : H^{1,1}(A) \rightarrow H^{1,1}(S),$$

denote by K the kernel of a^* . Then clearly $h^{1,1} \geq q^2 - \dim K$ and our strategy for proving Theorem 1.1 is to give an upper bound for the dimension of K .

Complex conjugation of forms gives a real structure to the vector spaces $H^{1,1}(A)$ and $H^{1,1}(S)$ and a^* is a real map; denote by $a_{\mathbb{R}}^*$ the induced map:

$$a_{\mathbb{R}}^* : H^{1,1}(A)_{\mathbb{R}} \rightarrow H^{1,1}(S)_{\mathbb{R}}.$$

Since $K(\mathbb{R}) = \ker(a_{\mathbb{R}}^*)$ we have $\dim_{\mathbb{C}} K = \dim_{\mathbb{R}} K(\mathbb{R})$. As in [5], we identify $H^{1,1}(A)$ with the vector space $M := M(q, \mathbb{C})$ of $q \times q$ complex matrices and $H^{1,1}(A)_{\mathbb{R}}$ with the real space $\mathcal{H} \subset M$ of the hermitian matrices. Remark that the real structure of M is here given by the involution $B \rightarrow {}^t \bar{B}$, and $\mathcal{H} = M(\mathbb{R})$. In this way we identify $K(\mathbb{R})$ with a real subspace of \mathcal{H} .

Definition 2.1. *For any $X \in \mathcal{H}$ the minimal inertia of X is the integer $m_X = \min\{n_+, n_-\}$ where (n_+, n_-) is the signature (or inertia) of X .*

Remark that $m_X = 0$ if and only if X is semidefinite, that for any non-zero real number λ we have $m_X = m_{\lambda X}$ and also that the rank of X is $\geq 2m_X$.

The following statement (which obviously generalizes to higher dimensions, see [5]) is an essential ingredient for studying the dimension of K :

Proposition 2.2. *If S has no irrational pencils of genus ≥ 2 then $m_X > 1$ for any $X \in K(\mathbb{R}) \setminus \{0\}$.*

Proof. Suppose $\omega = i \sum_{s,k} x_{k,s} dz_s \wedge d\bar{z}_k \in K(\mathbb{R}) \setminus \{0\}$, with $X = (x_{k,s})$ a hermitian matrix. Let as above (n_+, n_-) be the signature of X and assume by contradiction that $m_X = \min\{n_+, n_-\} \leq 1$; note that we may assume without restrictions (by taking $-\omega$, if necessary) that $n_- \leq 1$.

By diagonalizing X we may write $\omega = i(\sum_j \beta_j \wedge \bar{\beta}_j - \lambda \alpha \wedge \bar{\alpha})$ where α is a non zero $(1,0)$ -form, β_j are independent $(1,0)$ -forms on A , and $\lambda \in \mathbb{R}$, $\lambda \geq 0$. Note that $\lambda = 0$ gives the case $n_- = 0$.

The pullback via the Albanese map gives an identification of $H^{1,0}(A)$ with $H^{1,0}(S)$ and, by abuse of language, we set $a^*(\gamma) = \gamma$ for any $\gamma \in H^{1,0}(A)$. Since $a^*(\omega) = 0$, by integration on S , we obtain:

$$0 = \int_S a^*(\omega) \wedge \alpha \wedge \bar{\alpha} = i \sum_j \int_S \beta_j \wedge \bar{\beta}_j \wedge \alpha \wedge \bar{\alpha}.$$

Since all the summands have the same sign it follows that

$$0 = i \int_S \beta_j \wedge \bar{\beta}_j \wedge \alpha \wedge \bar{\alpha} = i \int_S \alpha \wedge \beta_j \wedge \overline{\alpha \wedge \beta_j}, \quad \text{for any } j.$$

This gives by positivity $\alpha \wedge \beta_j = 0$. Since, by hypothesis, S has no irrational pencils of genus ≥ 2 , the Castelnuovo - de Franchis theorem ([8], see also [4]) gives $\alpha = \beta_j = 0$. Hence $\omega = 0$, a contradiction. \square

3. PROOF OF PROPOSITION 1.2 AND THEOREM 1.1

Let M be the space of $q \times q$ complex matrices and $\mathcal{H} \subset M$ the real space of the hermitian matrices. Let $M_2 \subset M$ be the locus of the matrices of rank ≤ 2 and $\mathcal{H}_2 = M_2(\mathbb{R}) \subset \mathcal{H}$ be the set of hermitian matrices of rank ≤ 2 .

Consider the projective space $\mathbb{P} := \mathbb{P}(M)$ which is a complex projective space of dimension $N = q^2 - 1$. Let $\mathbb{P}_{\mathbb{R}} \subset \mathbb{P}$ be the real projective space corresponding to \mathcal{H} and $D_2 \subset \mathbb{P}$ the determinantal variety corresponding to M_2 :

$$D_2 = \{\langle X \rangle \in \mathbb{P} : X \in M_2\}.$$

Then we have

$$D_2(\mathbb{R}) = D_2 \cap \mathbb{P}_{\mathbb{R}} = \{\langle X \rangle \in \mathbb{P} : X \in \mathcal{H}_2\};$$

Moreover, $D_2(\mathbb{R})$ is the union of two components D_1 and D_0 , where D_1 is the closure of the locus of matrices with signature $(1, 1)$, that is of $\{\langle X \rangle \in D_2 : m_X = 1\}$, and $D_0 = \{\langle X \rangle \in D_2 : m_X = 0\}$. Note that the intersection $D_1 \cap D_0$ corresponds to the hermitian matrices of rank 1.

One has $\dim_{\mathbb{C}} D_2 = \dim_{\mathbb{R}} D_2(\mathbb{R}) = 4q - 5$ and moreover, by [9], and [10]:

Proposition 3.1. *The degree of D_2 is odd if and only if $q = 2^k + 1$.*

Proof. By [9], the degree of D_2 is

$$\deg D_2 = \prod_{j=0}^{q-3} \frac{\binom{q+j}{q-2}}{\binom{q-2+j}{q-2}} = \prod_{j=0}^{q-3} \frac{(q+j-1)(q+j)}{(j+1)(j+2)}$$

and by [10], sect. 6, this quantity is odd if and only if $q-2$ and $q-1$ have disjoint binary expansion, *i.e.* when $q = 2^k + 1$. \square

Let $h \in H^1(\mathbb{P}_{\mathbb{R}}, \mathbb{Z}_2)$ be the class of a hyperplane. We recall that the ring of \mathbb{Z}_2 -cohomology of the real projective space $\mathbb{P}_{\mathbb{R}}$ is generated by h with the relation $h^{N+1} = 0$, that is $H^*(\mathbb{P}_{\mathbb{R}}, \mathbb{Z}_2) \cong \mathbb{Z}_2[h]/h^{N+1}$. It follows that the dual \mathbb{Z}_2 -cohomology class $[Y]$ of any real algebraic variety $Y \subset \mathbb{P}_{\mathbb{R}}$ of dimension n and odd degree is not trivial, hence equal to h^{N-n} .

In particular, for $q = 2^k + 1$, by Proposition 3.1 the class $x = [D_2(\mathbb{R})]$ is

$$x = h^{N-(4q-5)} \in H^{N-(4q-5)}(\mathbb{P}_{\mathbb{R}}, \mathbb{Z}_2).$$

We assume from now on that $q = 2^k + 1$ and $q \geq 5$.

We now take into account the classes of the components of $D_2(\mathbb{R})$:

Lemma 3.2. *Let $y = [D_0]$ and $z = [D_1]$ in $H^{N-(4q-5)}(\mathbb{P}_{\mathbb{R}}, \mathbb{Z}_2)$. Then $y = 0$ and $z = x$.*

Proof. Recall that $x = y + z$. We want to prove that $y = 0$. Consider the hyperplane T in $\mathbb{P}_{\mathbb{R}}$ whose points are the trace zero matrices, $T := \{\langle X \rangle \in \mathbb{P}_{\mathbb{R}} : \text{trace}(X) = 0\}$. Then, since D_0 corresponds to semi-definite matrices, $T \cap D_0 = \emptyset$. This implies that the cup-product $[T] \cdot [D_0] = h \cdot y = 0$, hence $y = 0$ and consequently $z = x$. \square

Now consider the identity matrix I and the point $\mathbb{I} = \langle I \rangle \in \mathbb{P}_{\mathbb{R}} \subset \mathbb{P}$. We let C_2 be the cone over D_2 with vertex \mathbb{I} , $C_2 = \{\langle tX + sI \rangle : X \in M_2 \text{ and } \langle t : s \rangle \in \mathbb{P}^1(\mathbb{C})\}$.

Then C_2 is a variety of (complex) dimension $4q - 4$ invariant under conjugation in \mathbb{P} ; its real locus $C_2(\mathbb{R})$ is the union $C_1 \cup C_0$ of two real cones with vertices at \mathbb{I} over, respectively, D_1 and D_0 . It is easy to verify that any $\langle X \rangle \in C_1$ satisfies $m_X \leq 1$.

Furthermore, the degree of C_2 equals the degree of D_2 and therefore it is odd, for $q = 2^k + 1$.

Lemma 3.3. *The cohomology class $[C_1] \in H^{N-(4q-4)}(\mathbb{P}_{\mathbb{R}}, \mathbb{Z}_2)$ is non trivial, that is $[C_1] = h^{N-(4q-4)}$. In particular C_1 intersects any real projective subspace of codimension $4q - 4$.*

Proof. Write $[C_2(\mathbb{R})] = [C_0] + [C_1]$. Since the degree of C_2 is odd one has $[C_2(\mathbb{R})] \neq 0$. We note that the cohomology class associated to a cone over a cycle is zero if and only if the cycle is zero and so $[C_0] = 0$. \square

Proof of Proposition 1.2. Let $L \subset \mathcal{H}$ be a vector subspace such that every $X \in L \setminus \{0\}$ has minimal inertia $m_X > 1$. Denote by $\mathbb{L} \subset \mathbb{P}_{\mathbb{R}}$ its associate projective space and assume by contradiction that $c = \text{codim } L < 4q - 3$. It follows that $h^c \cdot [C_2] \neq 0$ and therefore $\mathbb{L} \cap C_1 \neq \emptyset$ by Lemma 3.3. This is a contradiction since for every $\langle X \rangle \in C_1$ we have $m_X \leq 1$. \square

Proof of Theorem 1.1. The statement follows immediately from the discussion in Section 2, Propositions 2.2 and Proposition 1.2. \square

4. APPLICATIONS

Theorem 1.1 has some applications. The first one concerns the still mysterious surfaces satisfying $p_g = 2q - 3$ (see [13]).

Proposition 4.1. *Let S be a minimal surface of general type without irregular pencils of genus ≥ 2 and satisfying $p_g = 2q - 3$. If $q = 2^k + 1, k \geq 3$, then the linear system $|K_S|$ has a fixed part.*

Proof. One has $\chi(S) = q - 2$ and so by Theorem 1.1, $K_S^2 < 8\chi(S)$. The result then follows by [13, Theorem 1.2] (cf. also [2]). \square

Remark 4.2. The only known examples of surfaces of general type without irregular pencils of genus ≥ 2 and satisfying $p_g = 2q - 3$ are the symmetric product of a genus 3 curve and the Schoen surface ([16]). By [15] no such surfaces exist for $q = 5$ and by [14] for $q \geq 6$ such surfaces will always have birational canonical map.

We can obtain also a lower bound for $h^{1,1}$ even when $q \neq 2^k + 1$:

Proposition 4.3. *Let S be a surface of general type without irregular pencils of genus ≥ 2 . If $q = 2^k + 1 + \epsilon$, with $0 < \epsilon < 2^k$, then $h^{1,1} \geq 4q - 3 - 4\epsilon$.*

Proof. Keep the notation of Section 2 and take a decomposition $\mathcal{H} = V \oplus W$, with $V = \{(x_{i,j}) \in \mathcal{H} : x_{i,j} = 0, \max(i,j) > 2^k + 1\}$ and W any complementary subspace; since V is naturally identified with the space of $(q - \epsilon) \times (q - \epsilon)$ hermitian matrices and $K(\mathbb{R}) \subset \mathcal{H}$, then $\dim K(\mathbb{R}) \leq \dim(K(\mathbb{R}) \cap V) + \dim W \leq [(q - \epsilon)^2 - (4(q - \epsilon) - 3)] + [q^2 - (q - \epsilon)^2] = q^2 - (4q - 3 - 4\epsilon)$. \square

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